

Meromorphic Lax representations of (1+1)-dimensional multi-Hamiltonian dispersionless systems

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Abstract

Rational Lax hierarchies introduced by Krichever are generalized. A systematic construction of infinite multi-Hamiltonian hierarchies and related conserved quantities is presented. The method is based on the classical R -matrix approach applied to Poisson algebras. A proof, that Poisson operators constructed near different points of Laurent expansion of Lax functions are equal, is given. All results are illustrated by several examples.

1 Introduction

First order PDE's of the form

$$(u_i)_t = \sum_j A_i^j(u)(u_j)_x \quad i, j = 1, \dots, n$$

are called hydrodynamic or dispersionless systems in (1+1)-dimension. An important subclass of such systems are these which have multi-Hamiltonian structure, infinite hierarchy of symmetries and conservation laws. Differential Poisson structures for hydrodynamic systems were introduced for the first time by Dubrovin and Novikov [1] in the form (1.1) with $c = 0$, where g^{ij} is a contravariant nondegenerate flat metric and Γ_k^{ij} are related coefficients of the contravariant Levi-Civita connection. Then, they were generalized by Mokhov and Ferapontov [2] to the nonlocal form

$$\pi_{ij} = g^{ij}(u)\partial_x - \sum_k \Gamma_k^{ij}(u)(u_k)_x + c(u_i)_x \partial_x^{-1}(u_j)_x \quad (1.1)$$

in the case when g^{ij} is of constant curvature c . The natural geometric setting of related bi-Hamiltonian structures (Poisson pencils) is the theory of Frobenius manifolds based on the geometry of pencils of contravariant metrics [3]. Nevertheless, the condition of nondegeneracy

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of g^{ij} for the above Poisson tensors is not necessary. The degenerate hydrodynamic Poisson tensors were considered by Grinberg [4] and Dorfmann [5].

In paper [6] Krichever introduced integrable dispersionless systems with rational Lax functions on \mathbb{CP}^1 of the form

$$L = p^N + \sum_{k=0}^{N-1} a_k p^k + \sum_{l=1}^{\alpha} \sum_{i=1}^{i_l} \frac{a_{l,i}}{(p - p_l)^i} \quad n \geq 0, \quad i_l \geq 0, \quad (1.2)$$

where a 's and the poles p_l are smooth dynamical fields. Then, around all poles of (1.2), i.e. ∞ and p_l , the powers of Laurent expansions of L generate infinite Lax hierarchies of commuting vector fields with Lie bracket being the canonical Poisson bracket (3.1) with $r = 0$. Moreover, near these poles one can construct infinite hierarchies of constants of motion. Rational Lax functions (1.2) with related Lax hierarchies have been introduced in [6] in the context of Whitham hierarchies and topological field theories. From this point of view they have been considered also in [7] and [8]. The bi-Hamiltonian structures of Benney and Toda like Lax hierarchies, but with Poisson bracket (3.1) with $r = 1$ and rational Lax functions was developed in [9]. Their various reductions were also studied. They also have been investigated in the context of degenerate Frobenius manifolds [10]. In [11], it was shown how to construct recursion operators for some classes of such rational Lax representations.

In the theory of nonlinear evolutionary PDE's (dynamical systems) one of the most important problems is a systematic construction of integrable systems. By integrable systems we understand those which have infinite hierarchy of commuting symmetries. It is well known that a very powerful tool, called the classical R -matrix formalism, can be used for systematic construction of (1+1)-dimensional field and lattice integrable dispersive systems (soliton systems) [12]-[16] as well as dispersionless integrable field systems [17]-[19]. Moreover, the R -matrix approach allows a construction of Hamiltonian structures and conserved quantities.

In this paper the systematic approach of classical R -matrices to (1+1)-integrable dispersionless multi-Hamiltonian systems with meromorphic Lax hierarchies is presented. In the frames of that formalism we generalize the results of Krichever onto a wider set of integrable hierarchies with rational Lax representations as well as we develop systematically their multi-Hamiltonian structures. Section 2 briefly presents a number of basic facts and definitions concerning the formalism of R -matrices on Poisson algebras. In section 3 we define Poisson algebras of meromorphic functions and construct R -matrices. We study multi-Hamiltonian structures and show the main theorem, that Poisson tensors constructed for fixed Poisson algebra at different points of Laurent expansions of L are equal and that related hierarchies mutually commute. In section 4 we investigate appropriate forms of meromorphic Lax functions, with finite number of dynamical fields, which permit construction of integrable dispersionless systems and illustrate results by a large number of examples.

2 Classical R -matrix theory on Poisson algebras

The crucial point of the formalism is the observation that integrable dynamics from some functions space can be represented by integrable dynamics from an appropriate Lie algebra in the form of Lax equation

$$L_t = \text{ad}_A^* L = [A, L], \quad (2.1)$$

i.e. a coadjoint action of some Lie algebra \mathfrak{g} on its dual \mathfrak{g}^* , with the Lax operators L taking values from this Lie algebra $\mathfrak{g}^* \cong \mathfrak{g}$, where $[\cdot, \cdot]$ is an appropriate Lie bracket. From (2.1) it is

clear that we confine to such algebras \mathfrak{g} for which its dual \mathfrak{g}^* can be identified with \mathfrak{g} through the duality map $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$. So, we assume the existence of a scalar product (\cdot, \cdot) on \mathfrak{g} which is symmetric, non-degenerate and ad-invariant: $(\text{ad}_a b, c)_{\mathfrak{g}} + (b, \text{ad}_a c)_{\mathfrak{g}} = 0$. This abstract representation (2.1) of integrable systems is referred to as the Lax dynamics. Obviously, we have one-to-one correspondence between given lax dynamics and original dynamics.

On the space of smooth functions on the dual algebra \mathfrak{g}^* there exists a natural Lie-Poisson bracket

$$\{H, F\}(L) := \langle L, [dF, dH] \rangle \quad L \in \mathfrak{g}^* \quad H, F \in \mathcal{C}^\infty(\mathfrak{g}^*), \quad (2.2)$$

where dF, dH are differentials belonging to \mathfrak{g} which can be calculated from

$$\left. \frac{d}{dt} F(L + tL') \right|_{t=0} = \langle L', dF(L) \rangle, \quad L, L' \in \mathfrak{g}^*. \quad (2.3)$$

A linear map $R : \mathfrak{g} \rightarrow \mathfrak{g}$, such that the bracket

$$[a, b]_R := [Ra, b] + [a, Rb] \quad (2.4)$$

is a second Lie product on \mathfrak{g} is called the classical R -matrix. We will additionally assume that R -matrices commute with derivatives with respect to evolution parameters, i.e.

$$(RL)_t = RL_t. \quad (2.5)$$

This property is equivalent to the assumption that R commutes with differentials of smooth maps from \mathfrak{g} to \mathfrak{g} . This property is used in the proof of Theorem 4.2 in [17], although not explicitly stressed there. The equality (2.5) will be used in subsection 3.5 to show a commutation between particular Lax hierarchies.

Definition 2.1 *Let \mathcal{A} be a commutative, associative algebra with unit. If there is a Lie bracket on \mathcal{A} such that for each element $a \in \mathcal{A}$, the operator $\text{ad}_a : b \mapsto \{a, b\}$ is a derivation of the multiplication, i.e. $\{a, bc\} = \{a, b\}c + b\{a, c\}$, then $(\mathcal{A}, \{\cdot, \cdot\})$ is called a Poisson algebra and bracket $\{\cdot, \cdot\}$ is a Poisson bracket.*

Thus, the Poisson algebras are Lie algebras with an additional structure.

Theorem 2.2 [17] *Let \mathcal{A} be a Poisson algebra with Poisson bracket $\{\cdot, \cdot\}$ and non-degenerate ad-invariant scalar product (\cdot, \cdot) with respect to which the operation of multiplication is symmetric, i.e. $(ab, c) = (a, bc)$, $\forall a, b, c \in \mathcal{A}$. Assume R is a classical R -matrix, such that (2.5) holds, then for each integer $n \geq 0$, the formula*

$$\{H, F\}_n = (L, \{R(L^n dF), dH\} + \{dF, R(L^n dH)\}) \quad (2.6)$$

where H, F are smooth functions on \mathcal{A} , defines a Poisson structure on \mathcal{A} . Moreover, all $\{\cdot, \cdot\}_n$ are compatible.

The related Poisson bivectors π_n , such that $\{H, F\}_n = (dF, \pi_n dH)$ are given by the following Poisson maps

$$\pi_n : dH \mapsto \{R(L^n dH), L\} + L^n R^* (\{dH, L\}), \quad n \geq 0 \quad (2.7)$$

where the adjoint of R is defined by the relation $(R^* a, b) = (a, Rb)$. Notice that the bracket (2.6) with $n = 0$ is just a Lie-Poisson bracket with respect to the Lie bracket (2.4). Referring

to the dependence on L , Poisson maps (2.7) are called linear for $n = 0$, quadratic for $n = 1$ and cubic for $n = 2$, respectively.

We will look for a natural set of functions in involution w.r.t. the Poisson brackets (2.6). Such functions are Casimir functions of the natural Lie-Poisson bracket (2.2). A sufficient condition for smooth function $F(L)$ to be a Casimir function is that its differential $dF \in \ker \text{ad}_L$, i.e. $[dF, L] = 0$. Hence, the following Lemma is valid

Lemma 2.3 [17] *Smooth functions on \mathcal{A} which are Casimir functions of the natural Lie-Poisson bracket (2.2) commute with respect to $\{\cdot, \cdot\}_n$. The Hamiltonian system generated by a Casimir function $C(L)$ and the Poisson structure $\{\cdot, \cdot\}_n$ is given by the Lax equation*

$$L_t = [R(L^n dC), L], \quad L \in \mathcal{A}. \quad (2.8)$$

Let us assume that an appropriate scalar product on Poisson algebra \mathcal{A} is given by the trace form $\text{Tr} : \mathcal{A} \rightarrow \mathbb{R}$, such that

$$(a, b) = \text{Tr}(ab).$$

As we have assumed a nondegenerate trace form Tr on \mathcal{A} , we will consider the most natural Casimir functionals given by the trace of powers of L , i.e.

$$dC_q(L) = L^q \iff \begin{cases} C_q(L) = \frac{1}{q+1} \text{Tr}(L^{q+1}) & \text{for } q \neq -1 \\ C_{-1}(L) = \text{Tr}(\ln L) & \text{for } q = -1 \end{cases} \quad (2.9)$$

for which the related gradients follows by (2.3). Then, taking these $C_q(L)$ as Hamiltonian functions, one finds a hierarchy of evolution equations which are multi-Hamiltonian dynamical systems

$$L_{t_q} = \{R(dC_q), L\} = \pi_0(dC_q) = \pi_1(dC_{q-1}) = \dots = \pi_l(dC_{q-l}) = \dots \quad (2.10)$$

For any R -matrix each two evolution equations in the hierarchy (2.10) commute due to the involutivity of the Casimir functions C_q . Each equation admits all the Casimir functions as a set of conserved quantities in involution. In this sense we will regard (2.10) as a hierarchy of integrable evolution equations.

To construct the simplest R -structure let us assume that the Poisson algebra \mathcal{A} can be split into a direct sum of Lie subalgebras \mathcal{A}_+ and \mathcal{A}_- , i.e. $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$, $[\mathcal{A}_\pm, \mathcal{A}_\pm] \subset \mathcal{A}_\pm$. Denoting the projections onto these subalgebras by P_\pm , the classical R -matrix is well defined as

$$R = \frac{1}{2}(P_+ - P_-) = P_+ - \frac{1}{2} = \frac{1}{2} - P_-. \quad (2.11)$$

Following the above scheme, we are able to construct in a systematic way integrable multi-Hamiltonian dispersionless systems, with infinite hierarchy of involutive constants of motion and infinite hierarchy of related commuting symmetries, on an appropriate Poisson algebras. Finally, in the last step, we reconstruct our multi-Hamiltonian hierarchies in the original function space of related dispersionless systems.

3 Lax hierarchies for dispersionless systems

3.1 Poisson algebras of meromorphic functions

Let \mathcal{F} be the algebra of meromorphic functions with a finite number of poles, i.e. these analytic functions which have no essential singularities, on a Riemann sphere \mathbb{CP}^1 (i.e. complex plane

with point at ∞). Let p be a point in \mathbb{CP}^1 . Assume now that this algebra depends effectively on an additional spatial variable $x \in \Omega$. Denote by \mathcal{A} the algebra of all smooth functions: $f : \Omega \rightarrow \mathcal{F}$, i.e. $\mathcal{A} = \mathcal{C}^\infty(\Omega, \mathcal{F})$. Let $\Omega = \mathbb{S}^1$ if we assume these functions to be periodic in x or $\Omega = \mathbb{R}$ if these functions supposed to belong to the Schwartz space for a fixed parameter p . The Poisson bracket on \mathcal{A} can be introduced in infinitely many ways as

$$\{f, g\}_r := p^r (\partial_p f \partial_x g - \partial_x f \partial_p g) \quad r \in \mathbb{Z} \quad f, g \in \mathcal{A}. \quad (3.1)$$

Then, fixing r , \mathcal{A} is the Poisson algebra with an appropriate bracket (3.1). Poisson brackets (3.1) are generalization of canonical Poisson bracket ($r = 0$) through the addition of p^r factor.

To construct classical R -matrices we have to decompose \mathcal{A} into a direct sum of Lie subalgebras. It can be done by expanding functions belonging to \mathcal{A} in an appropriate annulus near a given point λ . Three kind of points on parametrized by $x \in \mathbb{CP}^1$ will be important. Two fixed points: ∞ and 0 , as well as points being smooth fields $v(x)$ from Ω to \mathbb{CP}^1 .

3.2 Classical R -matrices

Once we fixed Poisson algebra we are able to construct R -matrices and related Lax vector fields for which the algebra \mathcal{A} constitutes the phase space.

The expansion around ∞ . First, let us consider the case of point at ∞ . Then, meromorphic functions from \mathcal{A} expanded around ∞ are given by Laurent series:

$$\mathcal{A}^\infty = \left\{ \sum_{i=-\infty}^N a_i(x) p^i \right\}, \quad (3.2)$$

where $a_i(x)$ are dynamical fields. To construct R -matrices we have to decompose \mathcal{A}_∞ into Lie subalgebras. For a fixed r let $\mathcal{A}_{\geq k-r}^\infty = \{\sum_{i \geq k-r} a_i(x) p^i\}$ and $\mathcal{A}_{< k-r}^\infty = \{\sum_{i < k-r} a_i(x) p^i\}$. Let ap^m and bp^n be elements from (3.2) of order m and n , respectively. Poisson bracket (3.1) between these elements has the order $m + n + r - 1$ as

$$\{ap^m, bp^n\}_r = (mab_x - na_x b) p^{m+n+r-1}.$$

Now, simple inspection shows that $\mathcal{A}_{\geq k-r}^\infty$ and $\mathcal{A}_{< k-r}^\infty$ are Lie subalgebras in the following cases:

1. $r = 0, k = 0$;
2. $r \in \mathbb{Z}, k = 1, 2$;
3. $r = 2, k = 3$.

So, fixing r we fix the Lie algebra structure with k numbering the R -matrices (2.11) given in the following form

$$R = P_{\geq k-r}^\infty - \frac{1}{2} \quad (3.3)$$

where $P_{\geq k-r}^\infty$ is an appropriate projection onto Lie subalgebra of functions expanded in Laurent series (3.2). So, Lax hierarchy (2.8) assigned by (3.3) for a given Lax function $L \in \mathcal{A}^\infty$ is

$$L_{t_q} = \left\{ \left(L^{\frac{q}{N}} \right)_{\geq k-r}^\infty, L \right\}_r \quad q \in \mathbb{Z}_+, \quad (3.4)$$

where $(\cdot)_{\geq k-r}^\infty \equiv P_{\geq k-r}^\infty(\cdot)$ and $N \neq 0$ is the highest order of L expanded in Laurent series at ∞ . So, if L has pole at ∞ then $N > 0$ and the powers are positive, or if L^{-1} has pole at ∞ then $N < 0$ and the powers are negative.

The expansion around 0. Meromorphic functions expanded near 0 constitute the following algebra

$$\mathcal{A}^0 = \left\{ \sum_{i=-m}^{\infty} a_i(x) p^i \right\}.$$

The situation here is similar to the previous case. So, R -matrices are defined for the same r and k as at ∞ and are of the form

$$R = \frac{1}{2} - P_{<k-r}^0.$$

Hence, Lax hierarchies are

$$L_{t_q} = - \left\{ \left(L^{\frac{q}{m}} \right)_{<k-r}^0, L \right\}_r \quad q \in \mathbb{Z}_+, \quad (3.5)$$

where $-m \neq 0$ is the lowest order of Laurent series of L expanded around 0. So, if L has pole at 0 then $m > 0$ and the powers are positive, while if L^{-1} has pole at 0 then $m < 0$ and the powers are negative.

Now, we will show that schemes for points at ∞ and at 0 are interrelated.

Proposition 3.1 *Under the transformation*

$$x' = x \quad p' = p^{-1} \quad t' = t$$

the Lax hierarchy (3.4) defined by $L \in \mathcal{A}^\infty$ for r, k transforms into the Lax hierarchy (3.5) defined by $L' = L \in \mathcal{A}^0$ for $k' = 3 - k, r' = 2 - r$, i.e.

$$L \text{ for } k, r \text{ at } \infty \iff L' = L \text{ for } k' = 3 - k, r' = 2 - r \text{ at } 0.$$

Proof. It follows from the observation that $\{\cdot, \cdot\}_r = p^r \partial_p \wedge \partial_x = -p^{2-r} \partial_{p'} \wedge \partial_{x'} = -\{\cdot, \cdot\}'_{r'}$ and $(L^n)_{\geq k-r}^\infty = (L'^n)_{\leq k-r}^0 = (L'^n)_{< k'-r'}^0$. \square

The expansion around $v(x)$. Now, we will consider meromorphic functions in the form of Laurent series expanded around some field $v(x)$:

$$\mathcal{A}^v = \left\{ \sum_{i=-m}^{\infty} a_i(x) (p - v(x))^i \right\}.$$

Notice, that $v(x)$ is a dynamical field of the same kind as coefficients $a_i(x)$. Let $\mathcal{A}_{\geq k-r}^v = \{\sum_{i \geq k-r} a_i(p - v)^i\}$ and $\mathcal{A}_{< k-r}^v = \{\sum_{i < k-r} a_i(p - v)^i\}$. Here the situation is a bit more complicated as one has to expand p^r in (3.1) at $v(x)$, i.e.

$$p^r = \sum_{s=0}^{\infty} \binom{r}{s} v(x)^{r-s} (p - v(x))^s$$

where $\binom{r}{s} = (-1)^s \binom{-r+s-1}{s}$ for $r < 0$. Hence, p^r as the element of \mathcal{A}_v , has the lowest order equal zero, the highest order equal r for $r \geq 0$ and infinity for $r < 0$. Therefore

$$\{a(p - v)^m, b(p - v)^n\}_r = ((p - v)^\alpha + \dots + v^r) \times (mab_x - na_xb) (p - v)^{m+n-1},$$

where $\alpha = r$ for $r \geq 0$ and α goes to ∞ for $r < 0$. One finds that $\mathcal{A}_{\geq k-r}^v, \mathcal{A}_{< k-r}^v$ are Lie subalgebras in the following cases:

1. $r = 0, k = 0, 1, 2;$
2. $r = 1, k = 1, 2;$
3. $r = 2, k = 2, 3$

and R -matrices have the form

$$R = \frac{1}{2} - P_{<k-r}^v. \quad (3.6)$$

However, we have to choose these R -matrices which commutes with derivatives with respect to evolution parameters. Let $L = \sum_i a_i(p-v)^i$. Then,

$$\begin{aligned} (RL)_t - RL_t &= P_{<k-r}^v L_t - (P_{<k-r}^v L)_t \\ &= P_{<k-r}^v \left(\sum_i (a_i)_t (p-v)^i - \sum_i i a_i v_t (p-v)^{i-1} \right) - \frac{d}{dt} \left(\sum_{i < k-r} a_i (p-v)^i \right) \\ &= (k-r) a_{k-r-1} v_t (p-v)^{k-r-1} \end{aligned}$$

and equality (2.5) holds when $k-r=0$. Hence, further on we will consider only R -matrices (3.6) for

$$k = r = 0, 1, 2.$$

In consequence, one finds the following Lax hierarchies related to R -matrices (3.6)

$$L_{t_q} = - \left\{ \left(L^{\frac{q}{m}} \right)_{<0}^v, L \right\}_r \quad q \in \mathbb{Z}_+ \quad r = 0, 1, 2,$$

where $-m \neq 0$ is the lowest order of Laurent series of L at v .

3.3 Scalar products

To construct Poisson structures one has to define an appropriate scalar product on \mathcal{A} . We will define it near a given point λ by means of the trace form in the algebra \mathcal{A} with the Poisson structure (3.1) for fixed r :

$$\begin{aligned} \text{Tr}_\infty f &= - \int_\Omega \text{res}_\infty (p^{-r} f) dx, \\ \text{Tr}_\lambda f &= \int_\Omega \text{res}_\lambda (p^{-r} f) dx, \quad \lambda = 0, v(x) \quad f \in \mathcal{A}, \end{aligned}$$

where res is the standard residue. In further considerations the residue theorem will be very useful. Let $f \in \mathcal{A}$ and Γ be a set of all finite poles of f . Then, according to the residue theorem

$$\sum_{i \in \Gamma} \text{res}_{\lambda_i} f = \frac{1}{2\pi i} \oint_{\gamma_\Gamma} f dp \equiv -\text{res}_\infty f \quad \lambda_i \neq \infty \quad (3.7)$$

where γ_Γ is closed curve encircling all finite poles of f . So, residue at ∞ may be different then zero even if f does not have singularity at this point.

Lemma 3.2 *For two arbitrary functions $f, g \in \mathcal{A}$ the scalar product:*

$$(f, g)_\lambda := \text{Tr}_\lambda(fg) \quad \lambda = \infty, 0, v(x) \quad (3.8)$$

is symmetric, nondegenerate and ad-invariant.

Proof. The nondegeneracy and symmetry of (3.8) are obvious. Let γ_λ be a closed curve circling once a finite pole λ , then

$$\begin{aligned} \text{Tr}_\lambda \{f, g\}_r &= \int_\Omega \text{res}_\lambda (\partial_p f \partial_x g) dx - \int_\Omega \text{res}_\lambda (\partial_x f \partial_p g) dx \\ &= \frac{1}{2\pi i} \int_\Omega \oint_{\gamma_\lambda} (\partial_p f \partial_x g) dp dx - \frac{1}{2\pi i} \int_\Omega \oint_{\gamma_\lambda} (\partial_x f \partial_p g) dp dx \\ &= \frac{1}{2\pi i} \int_\Omega \oint_{\gamma_\lambda} (\partial_x f \partial_p g) dp dx - \frac{1}{2\pi i} \int_\Omega \oint_{\gamma_\lambda} (\partial_x f \partial_p g) dp dx = 0, \end{aligned}$$

where we have integrated by parts with respect to p and x . Similar proof is for $\lambda = \infty$. Therefore

$$\begin{aligned} (\{f, g\}_r, h)_\lambda - (\{g, h\}_r, f)_\lambda &= \text{Tr}_\lambda (\{f, g\}_r h) - \text{Tr}_\lambda (\{g, h\}_r f) \\ &= \text{Tr}_\lambda (\{fh, g\}_r - f \{h, g\}_r) + \text{Tr}_\lambda (f \{h, g\}_r) = \text{Tr}_\lambda \{fh, g\}_r = 0, \end{aligned}$$

i.e. ad-invariance is proved. \square

For a given functional $H(L) \in \mathcal{C}^\infty(\mathcal{A})$ of $L \in \mathcal{A}$ the differential can be calculated by (2.3). But for functional $H = \int_\Omega h(u_i) dx$, where u_i are dynamical coefficient of $L \in \mathcal{A}$, we have to show how to construct dH . Differential of H constructed near a given point λ , will be denoted by $d_\lambda H \in \mathcal{A}$. Coefficients of $d_\lambda H$ depend on dynamical fields and usual variational derivatives $\frac{\delta H}{\delta u_i}$ in such a way that the trace duality assumes the usual Euclidean form, i.e.

$$(d_\lambda H, L_t)_\lambda = \text{Tr}_\lambda (d_\lambda H L_t) = \sum_i \int_\Omega \frac{\delta H}{\delta u_i} (u_i)_t dx. \quad (3.9)$$

Notice, that from (3.9) it follows that

$$\forall_{i,j} \quad (d_{\lambda_i} H, K)_{\lambda_i} = (d_{\lambda_j} H, K)_{\lambda_j} \quad (3.10)$$

where K is vector field on \mathcal{A} such that it spans exactly the same subspace of \mathcal{A} as L_t .

To find R^* , i.e. the adjoint operation to R , one has to determine the adjoint projections near λ from the following relation

$$\left((P^\lambda)^* f, g \right)_\lambda = (f, P^\lambda g)_\lambda \quad f, g \in \mathcal{A}^\lambda.$$

So, for 0 and ∞ we have

$$(P_{<k-r}^0)^* = 1 - P_{<2r-k}^0 \quad (P_{\geq k-r}^\infty)^* = 1 - P_{\geq 2r-k}^\infty.$$

The case of $\lambda = v(x)$ is more delicate. Let $A = \sum_m a_m(p-v)^m$ and $B = \sum_n b_n(p-v)^n$, then for $r \geq 0$:

$$\begin{aligned}
(A, P_{<0}^v B)_v &= \int_{\Omega} \text{res}_v \left(\sum_{s \geq 0} \sum_m \sum_{n < 0} \binom{-r}{s} v^{-r-s} a_m b_n (p-v)^{m+n+s} \right) dx \\
&= \int_{\Omega} \sum_{s \geq 0} \sum_{n < 0} \binom{-r}{s} v^{-r-s} a_{-n-s-1} b_n dx = \int_{\Omega} \sum_{s \geq 0} \sum_{m \geq -s} \binom{-r}{s} v^{-r-s} a_m b_{-m-s-1} dx \\
&= \int_{\Omega} \text{res}_v \left(\sum_{s \geq 0} \sum_{m \geq -s} \sum_n \binom{-r}{s} v^{-r-s} a_m b_n (p-v)^{m+n+s} \right) dx \\
&= \left(p^r \sum_{s \geq 0} \binom{-r}{s} v^{-r-s} (p-v)^s P_{\geq -s}^v A, B \right)_v,
\end{aligned}$$

where we used an appropriate expansion of p^{-r} at v . Hence

$$(P_{<0}^v)^* = 1 - p^r \sum_{s=0}^{\infty} \binom{-r}{s} v^{-r-s} (p-v)^s P_{<-s}^v \quad r \geq 0$$

for $r = 0$ it reduces to $(P_{<0}^v)^* = 1 - P_{<0}^v$. We will use simplified notation:

$$P'_v = \sum_{s=0}^{\infty} \binom{-r}{s} v^{-r-s} (p-v)^s P_{<-s}^v$$

as then $(P_{<0}^v)^* = 1 - p^r P'_v$.

3.4 Poisson structures

The Poisson structures (2.6) at respective points, related to respective R -matrices, are

$$\{H, F\}_{\lambda}^n = (d_{\lambda} F, \pi_{\lambda}^n d_{\lambda} H)_{\lambda} \quad \lambda = \infty, 0, v(x) \quad n \geq 0$$

for which Poisson operators are given by the following forms

$$\begin{aligned}
\pi_{\infty}^n d_{\infty} H &= \{ (L^n d_{\infty} H)_{\geq k-r}^{\infty}, L \}_r - L^n (\{ d_{\infty} H, L \}_r)_{\geq 2r-k}^{\infty}, \\
\pi_0^n d_0 H &= \{ L, (L^n d_0 H)_{< k-r}^0 \}_r - L^n (\{ L, d_0 H \}_r)_{< 2r-k}^0, \\
\pi_v^n d_v H &= \{ L, (L^n d_v H)_{< 0}^v \}_r - L^n p^r P'_v (\{ L, d_v H \}_r).
\end{aligned} \tag{3.11}$$

It is important here to mention that for a given Lax operator L it may happen that L_t does not span a proper subspace of the full Poisson algebra \mathcal{A} , i.e. the image of the Poisson operator $\pi^n dH$ does not coincide with this subspace. Then, in general, the Dirac reduction can be invoked for restriction of a given Poisson tensor to a suitable subspace.

Lemma 3.3 *The following relations will be needed to prove forthcoming theorem:*

$$\begin{aligned}
(d_{\infty} F, \{ L, (L^n d_0 H)_{< k-r}^0 \}_r)_{\infty} &= \left(d_0 H, L^n (\{ d_{\infty} F, L \}_r)_{\geq 2r-k}^{\infty} \right)_0, \\
(d_{\infty} F, L^n (\{ L, d_0 H \}_r)_{< 2r-k})_{\infty} &= \left(d_0 H, \{ (L^n d_{\infty} F)_{\geq k-r}^{\infty}, L \}_r \right)_0,
\end{aligned}$$

for arbitrary k and r , and

$$(d_\infty F, \{L, (L^n d_v H)_{<0}^v\}_r)_\infty = \left(d_v H, L^n (\{d_\infty F, L\}_r)_\infty^\infty \right)_v, \quad (3.12)$$

$$(d_\infty F, L^n p^r P'_v (\{L, d_v H\}_r))_\infty = \left(d_v H, \{ (L^n d_\infty F)_{\geq 0}^\infty, L \}_r \right)_v, \quad (3.13)$$

where $r \geq 0$.

Proof. We will prove only the first and last relations as for the two remaining ones the proof is similar. We use property of ad-invariance and we omit (or add) these elements which do not contribute in calculations of residues:

$$\begin{aligned} (d_\infty F, \{L, (L^n d_0 H)_{<k-r}^0\}_r)_\infty &= ((L^n d_0 H)_{<k-r}^0, \{d_\infty F, L\}_r)_\infty \\ &= \int_\Omega \text{res}_\infty (p^{-r} (L^n d_0 H)_{<k-r}^0 \{L, d_\infty F\}_r) dx = \int_\Omega \text{res}_\infty (p^{-r} (L^n d_0 H)_{<k-r}^0 (\{L, d_\infty F\}_r)_{\geq 2r-k}^\infty) dx \\ &\stackrel{\text{by (3.7)}}{=} \int_\Omega \text{res}_0 (p^{-r} (L^n d_0 H)_{<k-r}^0 (\{d_\infty F, L\}_r)_{\geq 2r-k}^\infty) dx \\ &= \int_\Omega \text{res}_0 (p^{-r} L^n d_0 H (\{d_\infty F, L\}_r)_{\geq 2r-k}^\infty) dx = \left(d_0 H, L^n (\{d_\infty F, L\}_r)_{\geq 2r-k}^\infty \right)_0. \end{aligned}$$

Let $r \geq 0$. Using proper expansion of p^{-r} at v we have:

$$\begin{aligned} (d_\infty F, L^n p^r P'_v (\{L, d_v H\}_r))_\infty &= - \int_\Omega \text{res}_\infty (d_\infty F L^n P'_v (\{L, d_v H\}_r)) dx \\ &= \int_\Omega \text{res}_\infty ((L^n d_\infty F)_{\geq 0}^\infty P'_v (\{d_v H, L\}_r)) dx \stackrel{\text{by (3.7)}}{=} \int_\Omega \text{res}_v ((L^n d_\infty F)_{\geq 0}^\infty P'_v (\{L, d_v H\}_r)) dx \\ &= \int_\Omega \text{res}_v ((L^n d_\infty F)_{\geq 0}^\infty \{L, d_v H\}_r) dx = ((L^n d_\infty F)_{\geq 0}^\infty, \{L, d_v H\}_r)_v \\ &= \left(d_v H, \{ (L^n d_\infty F)_{\geq 0}^\infty, L \}_r \right)_v. \end{aligned}$$

Thus all relations are valid. \square

Theorem 3.4 *Let $L \in \mathcal{A}$ be a meromorphic Lax function. Then for all appropriate k and r*

$$\{H, F\}_0^n = \{H, F\}_\infty^n \quad \text{and} \quad \pi_0^n d_0 H = \pi_\infty^n d_\infty H$$

while for $k = r = 0, 1, 2$

$$\forall_i \{H, F\}_{v_i}^n = \{H, F\}_\infty^n \quad \text{and} \quad \pi_{v_i}^n d_{v_i} H = \pi_\infty^n d_\infty H,$$

where v_i are dynamical fields. Therefore, Poisson structures, from the original function space of related dispersionless systems, calculated for fixed r and k at different points are equal.

Proof. We will prove only the second set of relations as for the first part the proof is similar. Thus,

$$\begin{aligned}
\{H, F\}_{v_i}^n &= (d_{v_i} F, \pi_{v_i}^n d_{v_i} H)_{v_i} \stackrel{\text{by (3.10)}}{=} (d_\infty F, \pi_{v_i}^n d_{v_i} H)_\infty \\
&= (d_\infty F, \{L, (L^n d_{v_i} H)_{<0}^{v_i}\}_r - L^n p^r P'_{v_i}(\{L, d_{v_i} H\}_r))_\infty \\
&\stackrel{\text{by (3.12-3.13)}}{=} (d_{v_i} H, L^n (\{d_\infty F, L\}_r)_\infty^\infty - \{(L^n d_\infty F)_{\geq 0}^\infty, L\}_r)_\infty \\
&= -(d_{v_i} H, \pi_\infty^n d_\infty F)_{v_i, r} \stackrel{\text{by (3.10)}}{=} -(d_\infty H, \pi_\infty^n d_\infty F)_\infty = \{H, F\}_\infty^n.
\end{aligned}$$

Now, from the equality of above Poisson brackets it follows that

$$(d_\infty F, \pi_\infty^n d_\infty H)_\infty = (d_{\lambda_i} F, \pi_{\lambda_i}^n d_{\lambda_i} H)_{\lambda_i} \stackrel{\text{by (3.10)}}{=} (d_\infty F, \pi_{\lambda_i}^n d_{\lambda_i} H)_\infty \iff \pi_{\lambda_i}^n d_{\lambda_i} H = \pi_\infty^n d_\infty H,$$

where $\lambda_i = 0, v_i$. Hence the theorem is proved. \square

3.5 Commuting multi-Hamiltonian Lax hierarchies

Let $L \in \mathcal{A}$ be a Lax function such that L and L^{-1} can have poles at $\infty, 0$ and $v_i(x)$. Then, for appropriate r and k near these poles one can construct the following multi-Hamiltonian Lax hierarchies (2.10)

$$L_{t_q} = \left\{ \left(L^{\frac{q}{N}} \right)_{\geq k-r}^\infty, L \right\}_r = \pi_\infty^0 d_\infty H_q^\infty = \pi_\infty^1 d_\infty H_{q-1}^\infty = \dots, \quad (3.14)$$

$$L_{\tau_q} = - \left\{ \left(L^{\frac{q}{m_0}} \right)_{< k-r}^0, L \right\}_r = \pi_0^0 d_0 H_q^0 = \pi_0^1 d_0 H_{q-1}^0 = \dots, \quad (3.15)$$

$$L_{\xi_q} = - \left\{ \left(L^{\frac{q}{m_i}} \right)_{< 0}^{v_i}, L \right\}_r = \pi_{v_i}^0 d_{v_i} H_q^{v_i} = \pi_{v_i}^1 d_{v_i} H_{q-1}^{v_i} = \dots, \quad (3.16)$$

where integer $q > 0$ and t, τ, ξ are evolution parameters. The Hamiltonians are then defined through trace forms near these poles and are given by (2.9) for $q \geq 0$

$$\begin{aligned}
H_q^\lambda(L) &= \frac{\epsilon}{\frac{q}{n} + 1} \int_\Omega \text{res}_\lambda \left(p^{-r} L^{\frac{q}{n} + 1} \right) \quad \text{for } q \neq -n \\
H_{-n}^\lambda(L) &= \epsilon \int_\Omega \text{res}_\lambda (p^{-r} \ln L) \quad \text{for } q = -n,
\end{aligned} \quad (3.17)$$

where $\epsilon = -1, n = N$ for $\lambda = \infty$ and $\epsilon = 1, n = m_0, m_i$ for $\lambda = 0, v_i$, respectively. Calculations of H_{-n}^λ from (3.17) for λ being the root of L may cause difficulties as then $\ln L$ has at λ essential singularity. There is an alternative approach. First we look for coefficients of dH_{-n}^λ which can be simply obtained from $\text{Tr}_\lambda(L^{-1} L_t) = \sum_i \int_\Omega \frac{\delta H_{-n}^\lambda}{\delta u_i}(u_i)_t dx$, since $d_\lambda H_{-n}^\lambda = L^{-1}$. Then, we calculate the functional H_{-n}^λ integrating a respective system of equations.

Let us show that Lax hierarchies (3.14-3.16) for fixed r and k mutually commute. Due to Lemma 2.3 Hamiltonians (3.17), as Casimirs of the natural Lie-Poisson bracket, are in involution with respect to Poisson brackets (3.4), i.e. $\{H_q^{\lambda_i}, H_{q'}^{\lambda_j}\}_{\lambda_k}^n = 0$, where $\lambda_i = \infty, 0, v_i$. From Theorem 3.4 it follows that $\pi_{\lambda_i}^n d_{\lambda_i} = \pi_{\lambda_j}^n d_{\lambda_j}$. Now, hence πd is the Lie algebra homomorphism,

from the algebra of smooth functions to the Lie algebra of vector fields, the commutation between Lax hierarchies (3.14-3.16) is immediate. For two vector fields $L_{t_q} = \pi_{\lambda_i}^n d_{\lambda_i} H_q^{\lambda_i}$ and $L_{t_{q'}} = \pi_{\lambda_j}^n d_{\lambda_j} H_{q'}^{\lambda_j}$ we have that

$$[L_{t_q}, L_{t_{q'}}] = [\pi_{\lambda_i}^n d_{\lambda_i} H_q^{\lambda_i}, \pi_{\lambda_j}^n d_{\lambda_j} H_{q'}^{\lambda_j}] = [\pi_{\lambda_i}^n d_{\lambda_i} H_q^{\lambda_i}, \pi_{\lambda_i}^n d_{\lambda_i} H_{q'}^{\lambda_j}] = \pi_{\lambda_i}^n d_{\lambda_i} \{H_q^{\lambda_i}, H_{q'}^{\lambda_j}\} = 0,$$

where $[\cdot, \cdot]$ is the Lie bracket between vector fields. However, for these commutations the Hamiltonian property is not necessary. We will show it for the Lax hierarchies (3.14) and (3.16) with $k = r = 0, 1, 2$ as for the other combinations the calculations are similar. We will use simplified notation $X = L^{\frac{q}{N}}, Y = L^{\frac{q'}{m_i}}$ and

$$(X)_{\geq 0}^\infty = X_{\geq 0}^\infty = X - X_{< 0}^\infty \quad (Y)_{< 0}^{v_i} = Y_{< 0}^v = Y - Y_{\geq 0}^v.$$

Then

$$\begin{aligned} & (L_{t_q})_{\xi_{q'}} - (L_{\xi_{q'}})_{t_q} = \\ &= \left\{ (X_{\geq 0}^\infty)_{\xi_{q'}}, L \right\}_r + \left\{ X_{\geq 0}^\infty, L_{\xi_{q'}} \right\}_r + \left\{ (Y_{< 0}^v)_{t_q}, L \right\}_r + \left\{ Y_{< 0}^v, L_{t_q} \right\}_r \\ & \stackrel{\text{by (2.5)}}{=} \left\{ (X_{\xi_{q'}})_{\geq 0}^\infty, L \right\}_r + \left\{ X_{\geq 0}^\infty, L_{\xi_{q'}} \right\}_r + \left\{ (Y_{t_q})_{< 0}^v, L \right\}_r + \left\{ Y_{< 0}^v, L_{t_q} \right\}_r \\ &= - \left\{ (Y_{< 0}^v, X)_r \right\}_{\geq 0}^\infty, L \Big\}_r - \left\{ X_{\geq 0}^\infty, \{Y_{< 0}^v, L\}_r \right\}_r \\ & \quad + \left\{ (\{X_{\geq 0}^\infty, Y\}_r)_{< 0}^v, L \right\}_r + \left\{ Y_{< 0}^v, \{X_{\geq 0}^\infty, L\}_r \right\}_r \\ &= \left\{ (\{X, Y_{< 0}^v\}_r)_{\geq 0}^\infty + (\{X_{\geq 0}^\infty, Y\}_r)_{< 0}^v - \{X_{\geq 0}^\infty, Y_{< 0}^v\}_r, L \right\}_r = 0 \end{aligned}$$

where we used Jacoby identity and the last equality holds since for $r = 0, 1, 2$

$$\begin{aligned} \{X_{\geq 0}^\infty, Y_{< 0}^v\}_r &= (\{X_{\geq 0}^\infty, Y_{< 0}^v\}_r)_{\geq 0}^\infty + (\{X_{\geq 0}^\infty, Y_{< 0}^v\}_r)_{< 0}^v \\ &= (\{X, Y_{< 0}^v\}_r)_{\geq 0}^\infty + (\{X_{\geq 0}^\infty, Y\}_r)_{< 0}^v. \end{aligned}$$

We see now that the restriction (2.5) is indeed crucial. Actually, in the same way one can prove commutations between symmetries inside these Lax hierarchies.

Combining results from current and previous subsections we obtain the following corollary.

Corollary 3.5 *Let L be a Lax function in \mathcal{A} with fixed Poisson bracket given by r and let us fix an appropriate k . Then, around each pole of L and L^{-1} one finds infinite hierarchy of commuting multi-Hamiltonian symmetries and infinite hierarchy of constants of motion. Moreover, vector fields from these different hierarchies mutually commute.*

In further considerations we are interested in extracting closed systems with finite number of dynamical functions. Therefore, we will look for meromorphic Lax functions, with finite number of dynamical coefficients, which allow a construction of consistent evolution Lax hierarchies. So, in the following section we will select an appropriate meromorphic Lax functions.

4 Meromorphic Lax functions

The meromorphic Lax function L is an appropriate one if the right-hand sides of Lax hierarchies (3.14-3.16) can be written in the form of evolutions L_t , i.e. left-hand sides. These Lax hierarchies are generated by positive and negative powers (in general fractional) of respective expansions near poles of L and L^{-1} . Actually, the appropriate expansions near ∞ and 0 are for $k = r = 0$; $k = 1, 2$ and $r \in \mathbb{Z}$; $k = 3, r = 2$, while the expansions near $v_i(x)$ takes place for $k = r = 0, 1, 2$. One finds these poles by looking for roots of L^{-1} and L , respectively. Important is the following. Let L be an appropriate Lax function with respect to the Lax hierarchy related to one of poles. Then, it is as well an appropriate function for hierarchies for all other allowed poles, for the same r and k . It is so, as by Proposition (3.4) the Lax hierarchy related to one pole can be rewritten for another one. Moreover, for a given L and fixed r and k , the Lax hierarchies, generated near all poles, will mutually commute.

We would like to investigate the general form of meromorphic Lax functions being appropriate Lax functions, i.e. such which allow a construction of integrable dispersionless equations. We will distinguish between three cases: the first one when L is a finite formal Laurent series at 0, the second one when L is a finite formal Laurent series at pole $v(x)$, and finally more general case of rational functions.

4.1 Polynomial Lax functions in p and p^{-1} .

Let us consider Lax functions of the form

$$L = u_N p^N + u_{N-1} p^{N-1} + \dots + u_{1-m} p^{1-m} + u_{-m} p^{-m}, \quad (4.1)$$

i.e. formal finite Laurent series at 0. The coefficients u_i are dynamical fields. For Lax functions (4.1), in general, we can construct powers near ∞ and 0 which will generate related Lax hierarchies (3.14) and (3.15), respectively. If $k = r$ powers calculated around roots of L generate additional Lax hierarchies given by (3.16).

From now on, without loss of generality, we will choose all appearing constants in the form that will simplify all formulae.

Proposition 4.1 *Lax function of the form (4.1) is an appropriate one in the following cases:*

1. $k = 0, r = 0$: $N \geq 2, u_N = 1, u_{N-1} = 0, m = 0$;
2. $k = 1, r \in \mathbb{Z}$: $N \neq 0, u_N = 1, m \neq 0$ for $r = 1$;
3. $k = 2, r \in \mathbb{Z}$: $N \neq 0$ for $r = 1, m \neq 0, u_{-m} = 1$;
4. $k = 3, r = 2$: $N = 0, m \geq 2, u_{1-m} = 0, u_{-m} = 1$.

We will not prove this proposition as it is the standard case considered in [19].

Proposition 4.2 *Under the transformation $p' = p^{-1}$ Lax hierarchies, from Proposition 4.1, generated by powers calculated at ∞ and 0 for appropriate r and k transforms into Lax hierarchies for 0 and ∞ with $r' = 2 - r$ and $k' = 3 - k$, respectively.*

The proof immediately follows from Proposition 3.1. Notice, that by transformation $p' = p^{-1}$ Lax hierarchies (3.16) for $r = k = 0, 1, 2$ defined at roots of L being dynamical fields fall out from the scheme presented in this article. On the other hand, for: $k = 1, r = 0$; $k = 2, r = 1$; $k = 3, r = 2$; according to Proposition 4.1 one can construct Lax hierarchies only at ∞ and 0. However, by $p' = p^{-1}$ they transform into cases: $k = 2, r = 2$; $k = 1, r = 1$; $k = 0, r = 0$; respectively, for which one is able to construct Lax hierarchies (3.16) related to all poles (including poles being dynamical fields) of L' and L'^{-1} . Hence, the relevant cases from Proposition 4.5 are:

- $k = 0, r = 0$;
- $k = 1, r \in \mathbb{Z} \setminus \{0\}$;
- $k = 2, r = 2$.

The remaining cases can be obtained by transformation $p' = p^{-1}$ according to Proposition 4.2.

To construct Poisson operators we have to choose a point near which we will perform the calculations. Nevertheless, as follows from Theorem 3.4, the explicit form of Poisson operators in the original function space is the same for all points. Thus, we choose the ∞ as it is the standard case. Then, as we assumed the usual Euclidean form (3.9), differentials of functional H are given by

$$dH \equiv d_\infty H = \sum_{i=-m}^{N+k-2} \frac{\delta H}{\delta u_i} p^{r-1-i},$$

where $m = 0$ for $k = 0$. Still we have to check whether the above Lax functions span proper subspaces, w.r.t. Poisson operators (3.11), of the full Poisson algebras. We will limit ourselves to linear ($n = 0$) and quadratic ($n = 1$) Poisson tensors, as obviously it is enough to define bi-Hamiltonian structures. Besides, in the all nontrivial cases Lax functions do not span proper subspaces w.r.t. Poisson tensors for $n \geq 2$.

Poisson tensors restricted to finite number of fields are properly defined if the highest and lowest orders of $\pi_\infty^n dH$ and L_t will coincide. Simple inspection shows that the highest order of $\pi_\infty^n dH$ is equal to $\max\{N + k - 2, nN + 2r - k - 1\}$ and the lowest is 0 for $k = 0$ and $\min\{k - 1 - m, -nm + 2r - k\}$ for $k = 1, 2$. Hence, in the case $k = 0$ the Lax function always span the proper subspace w.r.t. the linear Poisson tensor, but for $k = 1, 2$ only in case when $N \geq 2r - 2k + 1 \geq -m$, otherwise the Dirac reduction is required. The linear Poisson tensor is of the form

$$\pi_\infty^0 dH = \{(dH)_{\geq k-r}^\infty, L\}_r - (\{dH, L\}_r)_{\geq 2r-k}^\infty. \quad (4.2)$$

The reduced linear tensor for $N = -1$ and $k = r = 1, 2$ is given by (4.12). For the quadratic Poisson tensors the Dirac reduction is always necessary. The calculation procedure of Dirac reduction is explained in [16] (in a bit different notation). The reduced quadratic Poisson tensor for $k = r = 0, 1, 2$ is given by

$$(\pi_\infty^1)^{red} dH = \{(LdH)_{\geq 0}^\infty, L\}_r - L(\{dH, L\}_r)_{\geq r}^\infty + \frac{1}{N} \{L, \partial_x^{-1} \text{res}_\infty \{dH, L\}_0\}_r, \quad (4.3)$$

and for $k = 1, r = 0$ and $k = 2, r = 1$ takes the form

$$(\pi_\infty^1)^{red} dH = \{(LdH)_{\geq 1}^\infty, L\}_r - L(\{dH, L\}_r)_{\geq r-1}^\infty + \frac{1}{m} \{L, \partial_x^{-1} \text{res}_\infty \{dH, L\}_0\}_r. \quad (4.4)$$

Both reduced Poisson tensors are always local as $\text{res}_\infty\{\cdot, \cdot\}_0 = (\dots)_x$.

In the article, in general, we present examples for simplest Lax functions, where calculations are not very much complicated. From the Lax hierarchies considered we exhibit only the first nontrivial systems.

Example 4.3 *Two field system: $k = 1$, $r \in \mathbb{Z}$.*

Let us consider the Lax function of the form

$$L = p + u + vp^{-1}. \quad (4.5)$$

It has poles at ∞ and 0. Then, for ∞ we have

$$\begin{aligned} L_{t_{2-r}} &= \left\{ (L^{2-r})_{\geq 1-r}^\infty, L \right\}_r \iff \\ \left(\begin{matrix} u \\ v \end{matrix} \right)_{t_{2-r}} &= (2-r) \begin{pmatrix} (1-r)uu_x - v_x \\ -u_xv - (1-r)uv_x \end{pmatrix} = \pi_0 dH_{2-r}^\infty = \pi_1^{\text{red}} dH_{1-r}^\infty, \end{aligned} \quad (4.6)$$

where $(L^{2-r})_{\geq 1-r}^\infty = p^{2-r} + (2-r)up^{1-r}$. When $r = 2$ the next equation from the hierarchy is the first nontrivial one. For $r = 1$ this is the well known dispersionless Toda system. The hierarchy for 0 is the same as L has only two poles of the same order and $(L^q)_{\geq 1-r}^\infty = L - (L^q)_{< 1-r}^0$. The roots of L are $\lambda_\pm = \frac{1}{2}(-u \pm \sqrt{u^2 - 4v})$. Thus, for $r = 1$

$$(L^{-1})_{< 0}^{\lambda_\pm} = \frac{1}{1 - \frac{4v}{\lambda_\pm}} (p - \lambda_\pm)^{-1}$$

and one finds the following equations

$$\begin{aligned} L_{\xi_{-1}^\pm} &= - \left\{ (L^{-1})_{< 0}^{\lambda_\pm}, L \right\}_1 \iff \\ \left(\begin{matrix} u \\ v \end{matrix} \right)_{\xi_{-1}^\pm} &= \frac{\pm 1}{(u^2 - 4v)^{\frac{3}{2}}} \begin{pmatrix} 2u_xv - uv_x \\ v(2v_x - uu_x) \end{pmatrix} = \pi_0 dH_{-1}^{\lambda_\pm} = \pi_1^{\text{red}} dH_{-2}^{\lambda_\pm}. \end{aligned}$$

Of course, for $k = r = 1$ all equations mutually commute.

The Lax function (4.5) defines proper subspace w.r.t. the linear Poisson tensor (4.2) only for $r = 0, 1$. In the cases, the reduced quadratic Poisson tensors are given by (4.4) and (4.3), respectively. Hence, for $r = 0$

$$\pi_0 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \quad \pi_1^{\text{red}} = \begin{pmatrix} 2\partial & \partial u \\ u\partial & \partial v + v\partial \end{pmatrix}, \quad (4.7)$$

and related Hamiltonians are

$$H_1^\infty = \int_\Omega uv \, dx \quad H_2^\infty = \int_\Omega (u^2v + v^2) \, dx.$$

For $r = 1$

$$\pi_0 = \begin{pmatrix} 0 & \partial v \\ v\partial & 0 \end{pmatrix} \quad \pi_1^{\text{red}} = \begin{pmatrix} \partial v + v\partial & u\partial v \\ v\partial u & 2v\partial v \end{pmatrix}$$

and

$$\begin{aligned} H_0^\infty &= \int_\Omega uv \, dx & H_1^\infty &= \int_\Omega (u^2v + v^2) \, dx \\ H_{-2}^{\lambda_\pm} &= \int_\Omega \frac{\mp 1}{\sqrt{u^2 - 4v}} \, dx & H_{-1}^{\lambda_\pm} &= \pm \int_\Omega \ln \frac{u + \sqrt{u^2 - 4v}}{v} \, dx. \end{aligned}$$

Example 4.4 *Two field system: $k = 2, r = 2$.*

We will consider Lax function of the form

$$L = vp + u + p^{-1}$$

i.e. function (4.5) transformed by $p \mapsto p^{-1}$. By Proposition 4.2 the hierarchy for ∞ is given by hierarchy (4.6) for $k = 1, r = 0$ from above example. The roots of L are $\alpha_{\pm} = \frac{-u \pm \sqrt{u^2 - 4v}}{2v}$. Thus, for

$$(L^{-1})_{<0}^{\alpha_{\pm}} = \frac{1}{v - \frac{4v^2}{\alpha_{\pm}^2}} (p - \alpha_{\pm})^{-1}$$

one finds the following equations

$$\begin{aligned} L_{\xi_{-1}^{\pm}} &= - \left\{ (L^{-1})_{<0}^{\alpha_{\pm}}, L \right\}_2 \iff \\ \left(\begin{matrix} u \\ v \end{matrix} \right)_{t_{-1}} &= \frac{\pm 1}{(u^2 - 4v)^{\frac{3}{2}}} \begin{pmatrix} -uu_x + 2v_x \\ 2u_x v - uv_x \end{pmatrix} = \pi_0 dH_{-1}^{\alpha_{\pm}} = \pi_1^{red} dH_{-2}^{\alpha_{\pm}}. \end{aligned}$$

This system by Proposition 4.2 commutes with (4.6) for $r = 0$. Thus, the Poisson tensors are given by (4.7) with Hamiltonians

$$H_{-2}^{\alpha_{\pm}} = \int_{\Omega} \frac{\pm u}{2\sqrt{u^2 - 4v}} dx \quad H_{-1}^{\alpha_{\pm}} = \mp \int_{\Omega} \frac{1}{2} \left(u + \sqrt{u^2 - 4v} \right) dx.$$

4.2 Polynomial Lax functions in $(p - v)$ and $(p - v)^{-1}$

Let us consider Lax functions which are formal Laurent series around v , with a finite number of dynamical coefficients, of the form

$$L = u_N(p - v)^N + u_{N-1}(p - v)^{N-1} + \dots + u_{1-m}(p - v)^{1-m} + u_{-m}(p - v)^{-m} \quad m \neq 0. \quad (4.8)$$

Lax functions (4.8) have poles at ∞ and v , near which calculated powers generate, if allowed by k and r , respective Lax hierarchies. Additional powers with related hierarchies can be constructed around the roots of L .

Proposition 4.5 *Lax function of the form (4.8) is an appropriate one in the following cases:*

1. $k = r = 0$: $u_N = 1, u_{N-1} = Nv$;
2. $k = 1, r \in \mathbb{Z}$: $N \neq 0, u_N = 1, L|_{p=0} = 0$ for $r = 1$;
3. $k = 2, r \in \mathbb{Z}$: $N \neq 0$ when $r = 1, L|_{p=0} = 0, \frac{d}{dp}L \Big|_{p=0} = 1$;
4. $k = 3, r = 2$: $N = 0, L|_{p=0} = 0, \frac{d}{dp}L \Big|_{p=0} = 1, \frac{d^2}{dp^2}L \Big|_{p=0} = 0$.

Moreover, for the same k and r , the respective Lax hierarchies commute.

Proof. It is enough to consider the Lax hierarchy related to ∞ . Function (4.8) will be appropriate Lax function if the left- and right-hand sides of Lax hierarchy (3.14) will coincide and the number of independent equations will be the same as the number of dynamical coefficients in L . The Lax hierarchy (3.14) can be written in two equivalent representations

$$L_t = \{A_{\geq k-r}^\infty, L\}_r = -\{A_{< k-r}^\infty, L\}_r.$$

So, we have to examine expansions of this hierarchy near ∞ and v as well as at 0, since the factor p^r occur in Poisson bracket. It turns out that first representation yields direct access to terms with lowest orders, whereas the second representation yields information about terms with highest orders. Near ∞ we have

$$\begin{aligned} L_t &= (u_N)_t p^N + (u_{N-1} - Nv)_t p^{N-1} + \text{lower terms}, \\ L_t &= -\{A_{< k-r}^\infty, L\}_r = -\{\alpha p^{k-r-1} + l.t., u_N p^N + l.t.\}_r = (\dots)p^\alpha + l.t., \end{aligned}$$

where $\alpha = N + k - 2$ for $N \neq 0$ when $r = k - 1$; and $\alpha = 0$ for $N = 0$ and $r = k - 1$. This impose the constraints on fields u_N and u_{N-1} given in Proposition. The expansion of $A_{\geq k-r}^\infty$ near v , is of the form $A_{\geq k-r}^\infty = \text{higher terms} + \gamma_1(p - v) + \gamma_0$ as $A_{\geq k-r}^\infty$ does not have singularity at v . So, near v we have

$$\begin{aligned} L_t &= \text{higher terms} + (u_{-m} + (m-1)v)_t (p-v)^{-m} + mv_t (p-v)^{-m-1}, \\ L_t &= \{A_{\geq k-r}^\infty, L\}_r = \{h.t. + \gamma_0, h.t. + u_{-m}(p-v)^{-m}\}_r = h.t. + (\dots)(p-v)^{-m-1} \end{aligned}$$

and the lowest order of left- and right-hand side of (3.14) are always the same. The expansion of $A_{\geq k-r}^\infty$ near 0, is of the form $A_{\geq k-r}^\infty = \text{higher terms} + \gamma p^{k-r}$. So we have

$$\begin{aligned} L_t &= \text{higher terms} + \frac{1}{2} \left(\frac{d^2}{dp^2} L \Big|_{p=0} \right)_t p^2 + \left(\frac{d}{dp} L \Big|_{p=0} \right)_t p + \left(L \Big|_{p=0} \right)_t, \\ L_t &= \{A_{\geq k-r}^\infty, L\}_r = \left\{ h.t. + \gamma p^{k-r}, h.t. + \frac{d}{dp} L \Big|_{p=0} p + L \Big|_{p=0} \right\}_r = h.t. + (\dots)p^\alpha. \end{aligned}$$

For $k = r = 0$ we have $\alpha = 0$ and there is no need of additional constraints. For $k = 1$ if $r \neq 1$: $\alpha = 0$ and both sides have the same order in expansion at 0. But for $k = r = 1$ we have $\alpha = 1$. Hence, $(L|_{p=0})_t = 0$ and we have to impose the constraint $L|_{p=0} = 0$. Then, both sides have the same form. For $k = 2$ and arbitrary r : $\alpha > 0$ and the first constraint of the form $L|_{p=0} = 0$ is needed. Taking into consideration this constraint: $\alpha = 2$ and it follows that $(\frac{d}{dp} L|_{p=0})_t = 0$. Hence, both sides will agree if we impose an additional constraint $\frac{d}{dp} L|_{p=0} = 1$. For $k = 3$ the reasoning is similar to the case $k = 2$, but there will be one more constraint of the form $\frac{d^2}{dp^2} L|_{p=0} = 0$ needed. Commutation of Lax hierarchies follows from Corollary 3.5. \square

Proposition 4.6 *The case $k = r = 0$ of Proposition 4.5 by the transformation $p \mapsto p - v$ turns to the case $r = 0, k = 1$ of Proposition 4.1. Thus both Lax hierarchies are equivalent.*

Proof. Consider the transformation $p' = p - v, x' = x, t' = t$, where $t = t_q$ or ξ_q . Then, $\partial_p = \partial_{p'}, \partial_x = \partial_{x'} - v_x \partial_{p'}$ and $\partial_t = \partial_{t'} - v_t \partial_{p'}$. The points at ∞ and v transform into points at ∞ and 0, respectively and the Poisson bracket (3.1) for $r = 0$ is preserved:

$$\{\cdot, \cdot\}_0 = \partial_p \wedge \partial_x = \partial_{p'} \wedge (\partial_{x'} + v_{x'} \partial_{p'}) = \partial_{p'} \wedge \partial_{x'} = \{\cdot, \cdot\}'_0.$$

Let L be the Lax function of the form (4.8) from Proposition 4.5 for $r = k = 0$. Then, by the above transformation $L' = L$ is a Lax function of the form 4.1 from Proposition 4.5 for $r = 0, k = 1$. For meromorphic function $A \in \mathcal{A}$, let $(A)_0^\lambda$ mean the zero-order term of Laurent series at λ . From (3.14) and (3.16) it follows that

$$v_{t_q} = \left(\left(L^{\frac{q}{N}} \right)_0^\infty \right)_x \quad v_{\xi_q} = \left(\left(L^{\frac{q}{m}} \right)_0^v \right)_x,$$

respectively. Thus the left- and right-hand side of (3.16) are equal

$$\begin{aligned} L_{\xi_q} &= L'_{\xi'_q} - v_{\xi_q} L'_{p'} = L'_{\xi'_q} - \left(\left(L^{\frac{q}{m}} \right)_0^v \right)_x L'_{p'} = L'_{\xi'_q} + \left\{ \left(L'^{\frac{q}{m}} \right)_0^0, L' \right\}'_0, \\ L_{\xi_q} &= - \left\{ \left(L^{\frac{q}{m}} \right)_{<0}^v, L \right\}_0 = - \left\{ \left(L'^{\frac{q}{m}} \right)_{<0}^0, L' \right\}'_0. \end{aligned}$$

Hence,

$$L'_{\xi_q} = - \left\{ \left(L'^{\frac{q}{m}} \right)_{<1}^0, L' \right\}'_0.$$

Similar calculations are valid at ∞ . □

Notice that, for the case $k = r = 0$ of Proposition 4.5 one is able to construct Lax hierarchies related to the roots of L , which is not possible for the case $k = 1, r = 0$ of Proposition 4.1. In the sense, the first case is more general.

Proposition 4.7 *Under the transformation $p' = p^{-1}$, the following equalities between some cases from Proposition 4.5 hold:*

- *the Lax hierarchy related to 0 for $k = 3, r = 2$ is equivalent to the Lax hierarchy related to ∞ for $k = r = 0$ with $N = -1$;*
- *the Lax hierarchy related to 0 for $k = 2, r \neq 1$ with $N = 0$ is equivalent to the Lax hierarchy related to ∞ for $k = 1, r \neq 1$ with $N = -1$;*
- *Lax hierarchies related to ∞ and 0 for $k = 2, r = 1$ with $N = -1$ are equivalent to Lax hierarchies related to 0 and ∞ for $k = 1, r = 1$ with $N = -1$, $L|_{p=0} = 0$, respectively.*

Proof. The appropriate Lax function from Proposition 4.5 for $k = 3, r = 2$ has the form

$$L = u_0 + u_{-1}(p - v)^{-1} + \dots + u_{-m}(p - v)^{-m}$$

where $L|_{p=0} = 0$, $\frac{d}{dp}L|_{p=0} = 1$ and $\frac{d^2}{dp^2}L|_{p=0} = 0$. Taking into consideration the above constraints, expansion of L around 0 is $L = \dots + (\dots)p^2 + p$. By transformation $p' = p^{-1}$ we have that

$$(p - v)^{-1} = (p'^{-1} - v)^{-1} = -v' - v'^2(p' - v')^{-1}$$

where $v' = v^{-1}$. Thus L transforms into

$$L' = u'_0 + u'_{-1}(p' - v')^{-1} + \dots + u'_{-m}(p' - v')^{-m}.$$

From the expansion around 0 of L it follows that expansion of L' near ∞ is $L' = p'^{-1} + (\dots)p'^{-2} + \dots$. Hence, $u'_0 = 0$, $u'_{-1} = 1$ and the Lax function L' is an appropriate one for

$k = r = 0$. Analogously for two next relations in the proposition. The rest holds by Proposition 3.1. \square

Now, let us pass to the Hamiltonian formulation of Lax hierarchies related to the appropriate Lax functions from Proposition 4.5. In general, the relevant cases are for $k = 0, 1, 2$. Further we will consider only them. The differential at ∞ of functional H for the Lax function of the general form (4.8) is given by

$$dH \equiv d_\infty H = p^r \left(\frac{1}{mu_{-m}} \left(\frac{\delta H}{\delta v} + \sum_{i=1-m}^N iu_i \frac{\delta H}{\delta u_{i-1}} \right) (p-v)^m + \sum_{i=1-m}^{N+1} \frac{\delta H}{\delta u_{i-1}} (p-v)^{-i} \right) \quad (4.9)$$

as

$$\begin{aligned} \text{Tr}_\infty (L_t dH) &= - \int_\Omega \text{res}_\infty (p^{-r} L_t dH) dx \stackrel{\text{by (3.7)}}{=} \int_\Omega \text{res}_v (p^{-r} L_t dH) dx \\ &= \int_\Omega \left(\sum_{i=-m}^N (u_i)_t \frac{\delta H}{\delta u_i} + v_t \frac{\delta H}{\delta v} \right) dx. \end{aligned}$$

For the Lax functions with constraints from Proposition 4.5 one has to modify differentials (4.9) in an appropriate way or construct them by (4.11), i.e. the same as in the next subsection. One has to examine when a given Lax function from Proposition span the proper subspace with respect to Poisson tensors. The procedure is rather technical and similar to the proof of this proposition. Thus, we omit it and we will present only the final results. The Lax functions from Proposition 4.5 for $k = 0, 1, 2$ span proper subspace w.r.t. linear Poisson tensor $n = 0$ if $N \geq 2r - 2k + 1$, $m \geq -1$ and $r \geq k$. Then, it is given by (4.2). If it is not the case, Dirac reduction is required. The reduced linear Poisson tensor for $N = -1$, $m \geq 1$ and $k = r = 0, 1, 2$ is given by (4.12). These Lax functions do not form a proper subspace w.r.t. quadratic Poisson tensor $n = 1$ and always the Dirac reduction procedure is needed. For $k = r = 0, 1, 2$ reduced quadratic Poisson tensors have the form (4.3).

Example 4.8 *Two-field system: $k = r = 1$.*

The Lax function, taking into consideration appropriate constraints, is given by the form

$$L = (p - v) + u + v(u - v)(p - v)^{-1} = \frac{p(p + u - 2v)}{p - v}.$$

For ∞ one finds $(L)_{\geq 0}^\infty = p + u - v$ and the following equation

$$\begin{aligned} L_{t_1} &= \{ (L)_{\geq 0}^\infty, L \}_1 \iff \\ \begin{pmatrix} u \\ v \end{pmatrix}_{t_1} &= \begin{pmatrix} 2u_x v + uv_x - 2vv_x \\ u_x v \end{pmatrix} = \pi_0 dH_1^\infty = \pi_1^{\text{red}} dH_0^\infty. \end{aligned}$$

The Lax hierarchy related to v is the same as $L = (L)_{\geq 0}^\infty + (L)_{< 0}^v$. The Lax function has two roots 0 and $2v - u$. Then, for $(L^{-1})_{< 0}^0 = \frac{v}{2v - u} p^{-1}$ we have

$$L_{\tau_{-1}} = - \left\{ (L^{-1})_{< 0}^0, L \right\}_1 \iff \begin{pmatrix} u \\ v \end{pmatrix}_{\tau_{-1}} = \begin{pmatrix} \frac{v_x}{2v - u} \\ \frac{2vv_x - u_x v}{(u - 2v)^2} \end{pmatrix} = \pi_0 dH_{-1}^0 = \pi_1^{\text{red}} dH_{-2}^0.$$

The Lax hierarchy related to the root $2v - u$ is up to the sign the same as above since $L^{-1} = (L^{-1})_{\geq 0}^0 + (L^{-1})_{< 0}^{2v-u}$.

The general form for a differential of a given functional H according to (4.9) is

$$dH = \frac{(2-u)\frac{\delta H}{\delta u} + v\frac{\delta H}{\delta v}}{(u-v)v^2} p(p-v) + \frac{1}{v} \frac{\delta H}{\delta v} p.$$

The Lax function defines the proper subspace w.r.t. the linear Poisson tensor (4.2). The reduced quadratic Poisson tensors is given by (4.4). Then,

$$\pi_0 = \begin{pmatrix} \partial v + v\partial & \partial v \\ v\partial & 0 \end{pmatrix} \quad \pi_1^{red} = \begin{pmatrix} 2\partial uv + 2uv\partial & u\partial v + 2v\partial v \\ v\partial u + 2v\partial v & 2v\partial v \end{pmatrix}.$$

The respective Hamiltonians are

$$\begin{aligned} H_0^\infty &= \int_{\Omega} (u-v) dx & H_1^\infty &= \frac{1}{2} \int_{\Omega} (u^2 - v^2) dx \\ H_{-2}^0 &= \int_{\Omega} \frac{v-u}{(u-2v)^2} dx & H_{-1}^0 &= \int_{\Omega} \ln\left(\frac{u}{v} - 2\right) dx. \end{aligned}$$

4.3 Rational Lax functions

Let us consider the general form of meromorphic Lax function given by

$$L = \sum_{k=-m_0}^N u_k p^k + \sum_{i=1}^{\alpha} \sum_{k_i=1}^{m_i} a_{i,k_i} (p-v_i)^{-k_i} \quad (4.10)$$

where u_k , a_{i,k_i} and v_i are dynamical fields. From this class of functions considered in the following subsection we exclude those which have been examined earlier, i.e. (4.1) and (4.8). Any function (4.10) in general has a pole at ∞ of order N , at 0 of order m_0 and α evolution poles at v_j of order m_j . Then, one can construct positive powers of Laurent series at poles of L . Negative powers can be constructed as expansion at the roots of L . These powers generate for appropriate r and k Lax hierarchies (3.14-3.16).

Proposition 4.9 *Function of the form (4.10) is an appropriate one in the following cases:*

1. $k = r = 0$:

- $N \geq 1$, $u_N = 1$, $u_{N-1} = 0$, $m_0 = 0$,
- $\forall_k u_k = 0$, $\sum_{i=1}^{\alpha} a_{i,1} = 1$, $\sum_{i=1}^{\alpha} (a_{i,1}v_i + a_{i,2}) = 0$;

2. $k = 1$, $r \in \mathbb{Z}$:

- $N \geq 1$, $u_N = 1$, $m_0 \geq 1$,
- $N = -1$, $u_{-1} + \sum_{i=1}^{\alpha} a_{i,1} = 1$, $m_0 \geq 1$,
- $N \geq 1$, $u_N = 1$, $m_0 = 0$, $L|_{p=0} = 0$ for $r = 1$,
- $\forall_k u_k = 0$, $\sum_{i=1}^{\alpha} a_{i,1} = 1$, $L|_{p=0} = 0$ for $r = 1$;

3. $k = 2$, $r \in \mathbb{Z}$:

- $N \geq 1$, $m_0 \geq 1$, $u_{-m_0} = 1$,
- $N \geq 1$, $m_0 = 0$, $L|_{p=0} = 0$, $\frac{d}{dp}L|_{p=0} = 1$,
- $N = 0$, $u_0 = 0$ for $r = 1$, $m_0 \geq 1$, $u_{-m_0} = 1$,
- $\forall_{k \neq 0} u_k = 0$, $u_0 = 0$ for $r = 1$, $L|_{p=0} = 0$, $\frac{d}{dp}L|_{p=0} = 1$;

4. $k = 3$ and $r = 2$

- $N = 0$, $m_0 \geq 1$, $u_{1-m_0} = 0$, $u_{-m_0} = 1$,
- $\forall_{k \neq 0} u_k = 0$, $L|_{p=0} = 0$, $\frac{d}{dp}L|_{p=0} = 1$, $\frac{d^2}{dp^2}L|_{p=0} = 0$.

Moreover Lax hierarchies calculated at different points for the same r and k mutually commute. We excluded here the Lax functions of the form (4.1) and (4.8).

The proof is similar to the one of Proposition 4.5. The evolution of the general form of rational function (4.10) is given by

$$L_t = \sum_{k=-m_0}^N (u_k)_t p^k + \sum_{i=1}^{\alpha} \sum_{k_i=1}^{m_i} ((a_{i,k_i})_t + (k_i - 1)a_{i,k_i-1}(v_i)_t) (p-v_i)^{-k_i} + \sum_{i=1}^{\alpha} m_i a_{i,m_i}(v_i)_t (p-v_i)^{-m_i-1}.$$

The Lax hierarchies have to be examined near ∞ , 0 and dynamical poles. So, the meromorphic function of the form (4.10) will be an appropriate Lax function if:

- the right-hand sides of the Lax hierarchy considered and the time derivatives L_t will have the same order at all above poles,
- the number of independent equations, resulting from Lax hierarchies, will be the same as that of dynamical coefficients included in L .

The first condition implies all constraints considered in Proposition. To see that, an analysis like in proof of Proposition 4.5 is needed. So, by the first condition the right-hand sides of considered Lax hierarchies for appropriate r and k can be uniquely presented in the form of L_t , i.e. the left-hand sides. So, the second condition immediately follows from the first one.

The simplest way of deriving dispersionless systems related to a given meromorphic Lax functions is to transform Lax hierarchies into purely polynomial form in p through removal of finite singularities. It can be done by multiplication of both sides of Lax hierarchies by a proper factor.

Proposition 4.10 *Under transformation $p' = p^{-1}$ Lax hierarchies, from Proposition 4.9, defined at ∞ and 0 for appropriate r and k transforms into Lax hierarchies defined at 0 and ∞ for $r' = 2 - r$ and $k' = 3 - k$, respectively.*

See proof of Proposition 4.7 and the comment after Proposition 4.2.

Hence, the relevant cases from Proposition 4.9 are:

- $k = 0$, $r = 0$;
- $k = 1$, $r \in \mathbb{Z} \setminus \{0\}$;
- $k = 2$, $r = 2$.

The remaining cases can be obtained from the above cases by the transformation $p' = p^{-1}$ according to Proposition 4.10.

Once again we will consider Poisson tensors defined at ∞ . This time we are not going to present the explicit form of differentials $d_\infty H$ for the general meromorphic Lax function, but we will explain how to construct them. We postulate that

$$dH \equiv d_\infty H = \sum_{i=N_\infty-\beta+1}^{N_\infty} \gamma_i p^{r-i-1} \quad (4.11)$$

where β is a number of dynamical coefficients in L and N_∞ is the highest order of Laurent series of L_t at ∞ . The form (4.11) allows us to solve (3.9) ($\lambda = \infty$) to obtain functions γ_i in terms of dynamical coefficients of L and its variational derivatives such that we obtain the required Euclidean form. We will consider only relevant cases of meromorphic Lax functions from Proposition 4.9. Verification that they span the proper subspace with respect to Poisson tensors is similar to the proof of this proposition. These Lax functions span the proper subspace w.r.t. the linear Poisson tensor (4.2) for $k = 0$ if $N \geq 1$ and for $k = 1, 2$ if $N \geq 2r - 2k + 1 \geq -m_0$. If not the case, the Dirac reduction is required. The reduced linear tensors for $k = r = 0, 1, 2$ and $N = -1$ (N is the highest order of Laurent series of L at ∞) are given by

$$\begin{aligned} \pi_\infty^0 dH &= \{(dH)_{\geq 0}^\infty, L\}_r - (\{dH, L\}_r)_{\geq r}^\infty + \{\gamma_1 p + \gamma_0, L\}_r \\ \gamma_1 &= \partial_x^{-1} \text{res}_\infty \{dH, L\}_0 \\ \gamma_0 &= \partial_x^{-1} \text{res}_\infty \{dH, L\}_1 - \partial_x^{-1} (\gamma_1 ((L)_{-2}^\infty)_x + 2(\gamma_1)_x (L)_{-2}^\infty), \end{aligned} \quad (4.12)$$

where $(L)_{-2}^\infty$ is the coefficient staying at the term of order -2 in Laurent series at ∞ . For $k = r = 0$ we have $(L)_{-2}^\infty = 0$ and (4.12) simplified. Notice, that for $k = r = 0$ the reduced Poisson tensor (4.12) is always local, but for the remaining cases it is in general not. In the case of quadratic Poisson tensors, considered Lax functions do not span proper subspaces and the Dirac reduction is needed. The reduced quadratic Poisson tensor for $k = r = 0, 1, 2$ are given by (4.3), and for $k = 1, r = 0$ and $k = 2, r = 1$ by (4.4) where $m = m_0$.

Example 4.11 *The two-field system: $k = r = 0$.*

Let the Lax function have the form

$$L = u(p - v)^{-1} + (1 - u) \left(p - \frac{uv}{u - 1} \right)^{-1}.$$

The roots of L are ∞ and $\alpha = \frac{2uv-v}{u-1}$. Then, for ∞ we have

$$L_{t_2} = \left\{ (L^{-2})_{\geq 0}^\infty, L \right\}_0 \iff \left(\frac{u}{v} \right)_{t_2} = \left(\frac{2uv}{(3u-1)v^2} \right)_x = \pi_0^{\text{red}} dH_2^\infty = \pi_1^{\text{red}} dH_1^\infty,$$

where $(L^{-2})_{\geq 0}^\infty = p^2 + \frac{2uv^2}{u-1}$. The hierarchy for α is the same as $(L^q)_{\geq 0}^\infty = L - (L^q)_{< 0}^\alpha$. The function L has poles at v and $\lambda = \frac{uv}{u-1}$. At the point v one finds the following system

$$L_{\xi_1} = - \left\{ (L)_{< 0}^v, L \right\}_0 \iff \left(\frac{u}{v} \right)_{\xi_1} = \left(\frac{\frac{u(u-1)^3}{v^2}}{\frac{(u-1)^2}{v}} \right)_x = \pi_0^{\text{red}} dH_1^v = \pi_1^{\text{red}} dH_0^v,$$

where $(L)_{<0}^v = u(p-v)^{-1}$. The Lax function is invariant with respect to the transformation $u \mapsto 1-u$, $v \mapsto \frac{uv}{u-1}$. Therefore, the Lax hierarchy related to λ can be obtained through this transformation.

In the case the differential of a given functional calculated by (4.11) is

$$dH = \left(\frac{2(u-1)^3}{v^3} \frac{\delta H}{\delta u} + \frac{(u-1)^2}{uv^2} \frac{\delta H}{\delta v} \right) p^3 + \left(\frac{3(u-1)^2(2u-1)}{v^2} \frac{\delta H}{\delta u} + \frac{(u-1)(3u-1)}{uv} \frac{\delta H}{\delta v} \right) p^2.$$

Then, from (4.12) and (4.3) we find the following Poisson tensors

$$\pi_0^{red} = \begin{pmatrix} 0 & \partial(1-u) \\ (1-u)\partial & -\partial v - v\partial \end{pmatrix} \quad \pi_1^{red} = \begin{pmatrix} \partial \frac{u}{v^2}(u-1)^3 + \frac{u}{v^2}(u-1)^3 \partial & \frac{(u-1)^2}{v} \partial \\ \partial \frac{(u-1)^2}{v} & 0 \end{pmatrix},$$

respectively. The Hamiltonians are

$$H_1^\infty = \int_\Omega \frac{uv}{1-u} dx \quad H_2^\infty = \int_\Omega \frac{uv^2}{1-u} dx \quad H_0^v = \int_\Omega v dx \quad H_1^v = \int_\Omega \frac{u(u-1)^2}{v} dx.$$

Example 4.12 The four-field dispersionless system: $k = r = 0$.

For the Lax function of the form

$$L = p + a(p-v)^{-1} + b(p-w)^{-1}$$

near ∞ one finds

$$L_{t_2} = \left\{ (L^2)_{\geq 0}^\infty, L \right\}_0 \iff \begin{pmatrix} a \\ b \\ v \\ w \end{pmatrix}_{t_2} = \begin{pmatrix} 2av \\ 2bw \\ 2a + 2b + v^2 \\ 2a + 2b + w^2 \end{pmatrix}_x = \pi_0 dH_2^\infty = \pi_1^{red} dH_1^\infty,$$

where $(L^2)_{\geq 0}^\infty = p^2 + 2a + 2b$. Near to the v we have

$$L_{\xi_1} = - \left\{ (L)_{<0}^v, L \right\}_0 \iff \begin{pmatrix} a \\ b \\ v \\ w \end{pmatrix}_{t_1} = \begin{pmatrix} a - \frac{ab}{(v-w)^2} \\ \frac{ab}{(v-w)^2} \\ v + \frac{b}{v-w} \\ \frac{a}{v-w} \end{pmatrix}_x = \pi_0 dH_1^v = \pi_1^{red} dH_0^v,$$

where $(L)_{\leq 0}^v = a(p-v)^{-1}$. There are three, very complicated, roots of L . Thus, we are not going to calculate the respective equations.

The differential of a functional H is

$$dH = \left(\frac{2}{v-w} \frac{\delta H}{\delta b} + \frac{1}{b} \frac{\delta H}{\delta w} \right) \frac{(p-v)^2(p-w)}{(v-w)^2} + \left(\frac{2}{w-v} \frac{\delta H}{\delta a} + \frac{1}{a} \frac{\delta H}{\delta v} \right) \frac{(p-v)(p-w)^2}{(w-v)^2} \\ + \frac{1}{(v-w)^2} \frac{\delta H}{\delta b} (p-v)^2 + \frac{1}{(w-v)^2} \frac{\delta H}{\delta a} (p-w)^2.$$

Then, from (4.2) and (4.3) one finds the linear

$$\pi_0 = \begin{pmatrix} 0 & 0 & \partial & 0 \\ 0 & 0 & 0 & \partial \\ \partial & 0 & 0 & 0 \\ 0 & \partial & 0 & 0 \end{pmatrix}$$

and quadratic Poisson tensors

$$\pi_1^{red} = \begin{pmatrix} \partial a + a\partial - \frac{\partial ab}{(v-w)^2} - \frac{ab}{(v-w)^2}\partial & \frac{\partial ab}{(v-w)^2} + \frac{ab}{(v-w)^2}\partial & (v + \frac{b}{v-w})\partial & \frac{a}{v-w}\partial \\ \frac{\partial ab}{(v-w)^2} + \frac{ab}{(v-w)^2}\partial & \partial b + b\partial - \frac{\partial ab}{(v-w)^2} - \frac{ab}{(v-w)^2}\partial & -\frac{b}{v-w}\partial & (w - \frac{a}{v-w})\partial \\ \partial(v + \frac{b}{v-w}) & -\frac{\partial b}{v-w} & 2\partial & \partial \\ \frac{\partial a}{v-w} & \partial(w - \frac{a}{v-w}) & \partial & 2\partial \end{pmatrix},$$

respectively. The Hamiltonians are

$$\begin{aligned} H_1^\infty &= \int_\Omega (av + bw) \, dx & H_2^\infty &= \int_\Omega ((a+b)^2 + av^2 + bw^2) \, dx \\ H_0^v &= \int_\Omega a \, dx & H_1^v &= \int_\Omega \left(av + \frac{ab}{v-w} \right) \, dx. \end{aligned}$$

Example 4.13 *Four-field system: $k = 1$, $r \in \mathbb{Z}$.*

Let us consider a Lax function of the form

$$L = p + u + vp^{-1} + w(p-s)^{-1}. \quad (4.13)$$

It has poles at ∞ , 0 and w . Related equations to ∞ for $(L^{2-r})_{\geq 1-r}^\infty = p^{2-r} + (2-r)up^{1-r}$ are

$$\begin{aligned} L_{t_{2-r}} &= \left\{ (L^{2-r})_{\geq 1-r}^\infty, L \right\}_r \Longleftrightarrow \\ \begin{pmatrix} u \\ v \\ w \\ s \end{pmatrix}_{t_{2-r}} &= (2-r) \begin{pmatrix} (1-r)uu_x + v_x + w_x \\ u_xv + (1-r)uv_x \\ u_xw + (1-r)uw_x + (ws)_x \\ u_xs + (1-r)us_x + ss_x \end{pmatrix} = \pi_0 dH_{2-r}^\infty = \pi_1^{red} dH_{1-r}^\infty. \end{aligned}$$

The first equations from Lax hierarchies related to 0 for $r \neq 0$ are

$$L_{\tau_r} = - \left\{ (L^r)_{< 1-r}^0, L \right\}_r \Longleftrightarrow \begin{pmatrix} u \\ v \\ w \\ s \end{pmatrix}_{\tau_r} = rv^r \begin{pmatrix} \ln v \\ u - \frac{w}{s} \\ \frac{w}{s} \\ \ln \frac{s}{v} \end{pmatrix}_x = \pi_0 dH_r^0 = \pi_1^{red} dH_{r-1}^0,$$

where $(L^r)_{\leq 1-r}^0 = v^r p^{-r}$. But for $r = 0$ we have

$$L_{\tau_1} = - \left\{ (L)_{< 1}^0, L \right\}_0 \Longleftrightarrow \begin{pmatrix} u \\ v \\ w \\ s \end{pmatrix}_{\tau_1} = \begin{pmatrix} u - \frac{w}{s} \\ v - \frac{vw}{s^2} \\ \frac{vw}{s^2} \\ \frac{w-v}{s} - u \end{pmatrix}_x = \pi_0 dH_1^0 = \pi_1^{red} dH_0^0,$$

where $(L)_{\leq 1}^0 = u - \frac{w}{s} + vp^{-1}$. For $r = 1$ and $(L)_{\leq 0}^s = w(p-s)^{-1}$ one finds

$$L_{\xi_1} = - \left\{ (L)_{< 0}^s, L \right\}_1 \Longleftrightarrow \begin{pmatrix} u \\ v \\ w \\ s \end{pmatrix}_{\xi_1} = \begin{pmatrix} w_x \\ v \left(\frac{w}{s} \right)_x \\ u_xw + (ws)_x - v \left(\frac{w}{s} \right)_x \\ u_xs + ss_x + s \left(\frac{v}{s} \right)_x \end{pmatrix} = \pi_0 dH_1^s = \pi_1^{red} dH_0^s.$$

Once again we are not going to consider Lax hierarchies related to roots of L .

The differential of a functional H is

$$dH = \left(\frac{\delta H}{\delta s} + \frac{1}{s^2} \left(\frac{\delta H}{\delta v} - \frac{\delta H}{\delta w} \right) \right) p^{r+2} - \left(\frac{2}{s} \left(\frac{\delta H}{\delta v} - \frac{\delta H}{\delta w} \right) + \frac{1}{w} \frac{\delta H}{\delta s} \right) p^{r+1} + \frac{\delta H}{\delta v} p^r + \frac{\delta H}{\delta u} p^{r-1}.$$

The Lax function (4.13) span the proper subspace w.r.t linear Poisson tensor (4.2) only for $r = 0, 1$. The reduced quadratic tensors are for $r = 0, 1$ given by (4.4) and (4.3), respectively. Thus, for $r = 0$:

$$\pi_0 = \begin{pmatrix} 0 & \partial & 0 & 0 \\ \partial & 0 & 0 & -\partial \\ 0 & 0 & 0 & \partial \\ 0 & -\partial & \partial & 0 \end{pmatrix} \quad (4.14)$$

and

$$\pi_1^{red} = \begin{pmatrix} 2\partial & \partial(u - \frac{w}{s}) & \frac{\partial w}{s} & -\partial \\ (u - \frac{w}{s})\partial & \partial v(1 - \frac{w}{s^2}) - v(1 - \frac{w}{s^2})\partial & \frac{\partial vw}{s^2} + \frac{vw}{s^2}\partial & (\frac{w-v}{s} - u)\partial \\ \frac{w}{s}\partial & \frac{\partial vw}{s^2} + \frac{vw}{s^2}\partial & \partial w(1 - \frac{v}{s^2}) - w(1 - \frac{v}{s^2})\partial & (u + s + \frac{v-w}{s})\partial \\ -\partial & \partial(\frac{w-v}{s} - u) & \partial(u + s + \frac{v-w}{s}) & 2\partial \end{pmatrix}. \quad (4.15)$$

The related Hamiltonians are

$$\begin{aligned} H_1^\infty &= \int_{\Omega} (uv + uw + ws) \, dx & H_2^\infty &= \int_{\Omega} (u^2v + v^2 + ws^2 + 2uws + u^2w + 2vw + w^2) \, dx \\ H_0^0 &= \int_{\Omega} v \, dx & H_1^0 &= \int_{\Omega} v \left(u - \frac{w}{s} \right) \, dx. \end{aligned}$$

For $r = 1$ we have:

$$\pi_0 = \begin{pmatrix} 0 & \partial v & \partial w & \partial s \\ v\partial & 0 & 0 & 0 \\ w\partial & 0 & s\partial w + w\partial s & s\partial s \\ s\partial & 0 & s\partial s & 0 \end{pmatrix}$$

and

$$\pi_1^{red} = \begin{pmatrix} \partial(v+w) + (v+w)\partial & u\partial v & 2\partial ws + ws\partial + u\partial w & (u+s)\partial s \\ v\partial u & 2v\partial v & 2v\partial w & v\partial s \\ \partial ws + 2ws\partial + w\partial u & 2w\partial v & \pi_{ww} & (s^2 + us + v + 2w)\partial s \\ s\partial(u+s) & s\partial v & s\partial(s^2 + us + v + 2w) & 2s\partial s \end{pmatrix},$$

where $\pi_{ww} = \partial uws + uws\partial + w\partial(2s^2 + w) + (2s^2 + w)\partial w$. The related Hamiltonians are

$$\begin{aligned} H_0^\infty &= \int_{\Omega} u \, dx & H_1^\infty &= \int_{\Omega} \left(\frac{1}{2}u^2 + v + w \right) \, dx \\ H_0^0 &= \int_{\Omega} \left(u - \frac{w}{s} \right) \, dx & H_1^0 &= \int_{\Omega} \left(\frac{1}{2}u^2 + v - \frac{uw}{s} - \frac{vw}{s^2} + \frac{w^2}{2s^2} \right) \, dx \\ H_0^s &= \int_{\Omega} \frac{w}{s} \, dx & H_1^s &= \int_{\Omega} \left(w + \frac{uw}{s} + \frac{vw}{s^2} - \frac{w^2}{2s^2} \right) \, dx. \end{aligned}$$

Example 4.14 *Four-field system: $k = r = 2$.*

Lax function (4.13) transformed by $p \mapsto p^{-1}$ has the form

$$L = vp + u - \frac{w}{s} + p^{-1} - \frac{w}{s^2}(p - s^{-1})^{-1}.$$

For $(L)_{\leq 0}^{s^{-1}} = \frac{w}{s^2}(p - s^{-1})^{-1}$ one finds the system

$$L_{\xi_1} = - \left\{ (L)_{< 0}^{s^{-1}}, L \right\}_1 \Longleftrightarrow \begin{pmatrix} u \\ v \\ w \\ s \end{pmatrix}_{\xi_1} = \begin{pmatrix} -\frac{w}{s} \\ -\frac{vw}{s^2} \\ w \left(\frac{v}{s^2} - 1 \right) \\ -u - s + \frac{v-w}{s} \end{pmatrix}_x = \pi_0 dH_1^{s^{-1}} = \pi_1^{red} dH_0^{s^{-1}}$$

commuting, by Proposition (4.10), with equations from Example 4.13 for $r = 0$. The linear and quadratic Poisson tensors are given by (4.14) and (4.15), respectively. Hamiltonian functionals are given by

$$H_0^{s^{-1}} = - \int_{\Omega} w \, dx \quad H_1^{s^{-1}} = \int_{\Omega} w \left(u + \frac{v-1}{s} \right) \, dx.$$

5 Comments

In the article we have presented a systematic construction of multi-Hamiltonian dispersionless systems with meromorphic Lax representations. It is shown that for a given meromorphic Lax function L , if allowed by k and r , one can construct Lax hierarchies related to all poles of L and L^{-1} . These Lax hierarchies, if we fix k and r , mutually commute. It is shown how to construct Poisson tensors and infinite hierarchies of constants of motion. It is proved that Poisson tensors, from the original function space, reconstructed for different poles are equal. Also, we have examined systematically the forms of appropriate meromorphic Lax functions, with finite number of dynamical fields, allowing construction of consistent dispersionless systems. The Poisson tensors constructed for the appropriate meromorphic Lax functions considered in the following article are nondegenerate.

Articles [9]-[10] deal with rational Lax functions from the algebra with fixed Poisson bracket $r = 1$. However, only the Lax hierarchies generated by powers constructed near ∞ have been considered there. For the class of rational Lax functions used in these papers the bi-Hamiltonian structures are degenerate, i.e. the determinants of the related metrics vanish. The reason is that the constraint of the form $L|_{p=0} = 0$ is not taken into consideration. So, one dynamical field always can be represented as a function of all others. This fact entails the degeneracy of Poisson tensors.

There is a different approach to meromorphic Lax functions. From the complex analysis it is well known that meromorphic function can be uniquely presented in the factorized form. Because of such a factorization there is no problem in finding poles of L and L^{-1} near which one construct powers and related Lax hierarchies. Another advantage is that the dispersionless systems obtained have very symmetrical form. However, the disadvantage is that Poisson tensors are significantly more complicated. Such, factorized form of Lax functions as well allows for finding new reductions which are not obvious when we have Lax function in the standard form, see [9]-[10].

In the paper we have considered dispersionless systems with a finite number of dynamical fields. However, Lax function being infinite formal Laurent series leads to the construction

of dispersionless infinite-field Benney moment like equations. Such systems for Laurent series at ∞ have been considered earlier in [18]. The original Benney moment equation can be obtained for $k = r = 0$. If we consider formal Laurent series at a pole being a dynamical field $v(x)$ we will construct new classes of infinite-field dispersionless systems. They, together with bi-Hamiltonian structures, will be studied in a forthcoming article. Furthermore, all finite-field dispersionless systems, with meromorphic Lax functions, considered in this paper may be considered as reductions of these infinite-field systems.

All Lax functions used in the article belong to the algebras of meromorphic functions. But, it is straightforward to extend the theory presented into algebras of holomorphic functions. So, it may be worth looking systematically for new classes of appropriate Lax functions being holomorphic and allowing construction of related dispersionless systems.

Another issue is the extension of the theory of meromorphic Lax representations presented for dispersionless systems in order to construct integrable dispersive soliton systems for rational Lax operators. The first approach towards this was made in article [20]. However, the authors constructed soliton systems related only to the case $k = r = 0$ from our article. A more general theory, of dispersive deformations of formal Lax functions being polynomials in p and p^{-1} , is presented in our paper [16]. This approach is based on the Weyl-Moyal-like quantization procedure. The idea relies on the deformation of the usual multiplication in the algebra \mathcal{A} to the new associative but non-commutative \star -product. However, this theory works only for $r = 0, 1, 2$. Deformations of Poisson algebras for $r = 0, 2$ are equivalent and lead to the construction of field soliton systems, but for $r = 1$ they lead to the construction of lattice soliton systems. So, in a forthcoming article we are going to present a general theory of the field and lattice soliton systems for rational Lax operators.

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